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Halanay-type Theory in the Context of Evolutionary Equations with Time-lag

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Abstract

We consider extensions and modifications of a theory due to Halanay, and the context in which such results may be applied. Our emphasis is on a mathematical framework for Halanay-type analysis of problems with time lag and simulations using discrete versions or numerical formulae. We present selected (linear and nonlinear, discrete and continuous) results of Halanay type that can be used in the study of systems of evolutionary equations with various types of delayed argument, and the relevance and application of our results is illustrated, by reference to delay-differential equations, difference equations, and Θ-methods.

Key words:
Generalizations of Halanay's lemma, Difference/delay-differential inequalities with maxima, Applications, Deterministic & Itô delay equations, Θ-methods

1. EXTENDED INTRODUCTION

The results discussed here are related to evolutionary problems for a scalar or m-dimensional system of nonlinear functional differential equations with a possibly unbounded time lag. The class of problems includes delay differential equations (*DDEs* – §2). We give various extensions of Halanay's lemma that can be used to discuss the properties of DDEs and their preservation under discretization. In that context, we discuss DDEs with various types of time lag, various discrete analogues, and numerical (Θ-) methods, and give examples. We promote the view that sound computational analysis requires firm theoretical foundations, and our emphasis is on the mathematical infrastructure.

1.1. Halanay's lemma

To clarify our remarks, we give Halanay's original lemma, stated here in the original notation:

Proposition 1.1. (Halanay's Lemma [18, pp. 378–380])

If $\overset{\bullet}{f}(t) \le -\alpha f(t) + \beta \sup_{t-\tau\le\sigma\le t} f(\sigma)$ and if $\alpha > \beta > 0$, then there exists $\gamma > 0$ and $k > 0$ such that $f(t) \le k\exp\{-\gamma(t-t_0)\}$ for $t \ge t_0$.

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It is assumed that $\tau \geq 0$ and f is a scalar-valued positive function with derivative $\overset{\bullet}{f}$. Let $\mathcal{Q}(\zeta) = \zeta - \alpha + \beta \exp(\zeta\tau)$ (*cf.* Example 2.1). From the text of [18], we learn that

$$k = \sup_{t_0-\tau \leq \sigma \leq t_0} f(\sigma);\ \gamma \text{ is the real zero of } \mathcal{Q}(\zeta) \tag{1.1a}$$

ensures, for $0 < \beta \leq \alpha$, that

$$f(t) \leq k\mathrm{e}^{-\gamma(t-t_0)} \tag{1.1b}$$

($\gamma \in (0, \alpha - \beta]$ if $0 < \beta < \alpha$; $\gamma = 0$ if $\alpha = \beta$). We refer to the above as *Halanay's (original) Lemma*. The quality of the bound on $f(t)$ obtained using (1.1) can be judged by considering the case $f(t) = f(t_0)\exp\{-\gamma(t - t_0)\}$ ($t \in [t_0 - \tau, \infty)$). Halanay was interested in the effect of the delay term in a linear DDE when it was regarded as a perturbation of some ordinary differential equation (*ODE*).

1.2. Objectives and motivation

Halanay's theory can be extended in various directions: (*i*) by the production of modifications of Proposition 1.1; (*ii*) by novel applications of the theory. Under the heading (*i*) we may include converse results, results expressed in terms, say, of a one-sided derivative, results with more general history dependence than in Halanay's original lemma, nonlinear extensions, discrete analogues and (not addressed here) vector rather than scalar inequalities. Under (*ii*) we can include generalizations and analogues of DDEs to more general classes of causal (Volterra) equations including Volterra integro-differential equations.

The main interest that motivates the present author is *analysis of qualitative properties of solutions* of DDEs (also called retarded differential equations, *RDEs* [2]), and related problems, with a view to their preservation when simulating the solution using discrete versions. Our emphasis is the search for, and exploitation of, tools (tools linked to Halanay's work) for this task. His Lemma could be regarded as an application of a *comparison method.* We include material on Θ-methods (Example 2.2 and §2.6 *et seq.*) to demonstrate what type of tool is useful when discussing discretized schemes for DDEs. Note that discrete recurrences may be analyzed via 'densely defined extensions' (§2.3).

Some of our generalizations may be regarded as '*technical*', but relatively small changes can extend the scope of application. We are limited in the space available, so omit some proofs and do not explore periodicity, or various types of *numerical stability* associated [2, 10] with letters of the alphabet, such as G- $P-$stability; that would distract us from the emphasis we seek.

1.3. Notation for derivatives and generalizations

The notation $\psi[s', s''] := \{\psi(s'') - \psi(s')\}/\{s'' - s'\}$for a *divided difference* (where ψ is a real-valued function) is common in numerical analysis. If they exist, the *one-sided derivatives* of ψ at t are denoted by $\psi'_{\pm}(t)$ or $(\frac{\mathrm{d}}{\mathrm{d}t})_{\pm}\psi(t)$; *e.g.*, $\psi'_{+}(t) := \lim_{\delta\to 0} \psi[t + |\delta|, t]$ denotes the right-hand derivative, while $\psi'(t)$ denotes the conventional derivative. If ψ is continuous, it possesses the four Dini-derivatives $\mathrm{D}^{\pm}_{\pm}\psi(t) \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ where, in particular (see [22, p.7]),

$$\text{upper right Dini derivative:}\quad \mathrm{D}^{+}\psi(t) := \limsup_{\delta\to 0} \psi[t, t + |\delta|], \tag{1.2a}$$

$$\text{lower left Dini derivative:}\quad \mathrm{D}_{-}\psi(t) := \liminf_{\delta\to 0} \psi[t - |\delta|, t]. \tag{1.2b}$$

D, with no embellishment, denotes any (fixed) choice of Dini derivative.

Remark 1.1. *The upper and lower right Dini derivatives reduce to the right-hand derivative when it exists (and to the conventional derivative if it exists). For continuous $\psi_{1,2}$, $\mathrm{D}^{+}\{\psi_1(t) + \psi_2(t)\} \leq$*

$\mathrm{D}^+\psi_1(t) + \mathrm{D}^+\psi_2(t)$, *with equality if* $\mathrm{D}^+\psi_2(t) = (\frac{\mathrm{d}}{\mathrm{d}t})_+\psi_2(t)$. *The result (for any choice of Dini derivative* D*)*

$$\inf_{s_1,s_2\in[t',t'']} \psi[s_1,s_2] = \inf_{s\in[t',t'']} \mathrm{D}\psi(s), \qquad \sup_{s_1,s_2\in[t',t'']} \psi[s_1,s_2] = \sup_{s\in[t',t'']} \mathrm{D}\psi(s), \tag{1.3}$$

where ψ *is assumed continuous on* $[t',t'']$, *can be linked to a discussion of monotonicity. In a classroom note, Chalkey [11] gave a simple proof of (1.3).*

1.4. Halanay's preliminary results

Though Halanay's Lemma was referred to by Driver [13, p.390] (where it was afforded a succinct proof) and by Gopalsamy [15], Halanay's work was for some time under-exploited in western literature. (Halanay-type theory does not necessarily provide optimal results, but it provides an insight into the use of other theories and deserves to be more widely appreciated.) Without returning to the English-language source [18], readers may not appreciate that Halanay deduced his lemma as a corollary of a nonlinear result: Proposition 1.1 follows (*cf.* [18]) from Proposition 1.2(*b*) which itself follows from Proposition 1.2(*a*).

Assumption 1.1. *Let* $w\big(t,u,v\big)$ *be continuous for all* (u,v) *and* $t_0 \le t < \infty$, *and suppose it to be monotone-increasing with respect to* v.

Proposition 1.2. (Following [18]) *(a) If, with Assumption 1.1 and* $\tau \in \mathbb{R}_+$,

$$\mathrm{D}_- f_1(t) < w\big(t, f_1(t), \sup_{t-\tau\le s\le t} f_1(s)\big), \qquad \mathrm{D}_- f_2(t) \ge w\big(t, f_2(t), \sup_{t-\tau\le s\le t} f_2(s)\big),$$

and $f_1(s) < f_2(s)$ *for* $t_0 - \tau \le s \le t_0$, *then* $f_1(t) < f_2(t)$ *for* $t \in [t_0,\infty)$. *(b) Suppose that* $\mathrm{D}_- f(t) \le w\big(t, f(t), \sup_{t-\tau\le s\le t} f(s)\big)$ *for* $t_0 \le t < \infty$ *and, also, that the (upper) solution* $y(t)$ *of* $\dot{y}(t) = w\big(t, y(t), \sup_{t-\tau\le s\le t} y(s)\big)$ $(t \ge t_0)$, *with* $y(t) = f(t)$ *for* $t \in [t_0-\tau, t_0]$, *exists on* $[t_0,\infty)$. *Then* $f(t) \le y(t)$ *for* $t \ge t_0$.

Later, but not here, we ask that $w(t,0,0) = 0$. Where (in his statement of Proposition 1.2) Halanay used assumptions concerning a Dini derivative $\mathrm{D}_- f_{1,2}(t)$ or, in (*b*), $\mathrm{D}_- f(t)$, the result remains true if the condition is replaced by the corresponding condition on $\dot{f}_{1,2}(t)$ or, in (*b*), $\dot{f}(t)$. The use, in the literature, of different Dini derivatives may seem confusing: see [33] for insight.

1.5. Sample results extending Halanay's theory

Proposition 1.1 was stated above using Halanay's original notation. Instead of following this notation, in the remainder of this paper we use the notation p rather than f to represent an arbitrary positive function defined on $[t_*,\infty)$. Halanay's Lemma relates to a linear inequality and gives a bound. We sample a few of the ways in which Proposition 1.1 can be extended:

Ex.1.1. *Halanay's original lemma can be used to bound positive solutions of*

$$\mathrm{p}'(t) \le -\alpha\mathrm{p}(t) + \beta_0 \textstyle\int_{t-\tau_0}^t \mathrm{p}(s)\mathrm{d}s + \sum_{\ell=1}^L \beta_\ell \mathrm{p}(t-\tau_\ell) \ \textit{for}\ t \ge t_0. \tag{1.4}$$

Suppose α, $\{\beta_\ell\}_1^L$, *are all positive, with* $\sum_{\ell=0}^L \beta_\ell < \alpha$ *and* $\{\tau_\ell\}_0^L$ *are all positive and bounded above by* τ^+ *If the positive differentiable function* p *satisfies (1.4), it follows that* $\mathrm{p}'(t) \le -\alpha\mathrm{p}(t) + \beta\mathrm{p}(t-\tau_\natural(t))$ *where* $0 \le \tau_\natural(t) \le \tau^+$ *and* $\beta = \sum_{\ell=0}^L \beta_\ell$, *so Proposition 1.1 can be applied if* $0 < \beta < \alpha$ *using* γ *in (1.1). However, the value* γ *can be improved, by a direct investigation.*

Ex.1.2. *Suppose (to obtain a type of converse result) that* $\alpha(t) \geq \alpha_* > 0$, $\beta(t) \geq \beta_* \geq 0$, *for* $t \in [t_0, \infty)$, *where* α, β *are bounded and continuous on* $[t_0, \infty)$. *Suppose* $t - \tau(t) \leq t$, $t_* = \inf_{t\in[t_0,\infty)} t - \tau(t)$ *and* $t - \tau(t) \to \infty$ *as* $t \to \infty$ *and* p *is bounded, positive, and continuous on* $[t_*, t_0]$. If $\mathrm{p}(t) > 0$ satisfies $\mathrm{p}'(t) \geq \alpha(t)\mathrm{p}(t) - \beta(t) \sup_{t-\tau(t)\leq s\leq t} \mathrm{p}(s)$ (for $t \geq t_0$), and if there exists a value $\sigma > 0$ such that $\alpha(t) - \beta(t) \geq \sigma > 0$ for $t \geq t_0$, then there is a sequence $\{t^{[\ell]}\}_{\ell\in\mathbb{N}}$ with $\lim_{\ell\to\infty} t^{[\ell]} = \infty$ such that $\mathrm{p}(t^{[\ell]}) \to \infty$ as $t^{[\ell]} \to \infty$.

Ex.1.3. *Amongst others (see, for example, [24]), Tang [30] explored discrete analogues and extensions of Halanay's lemma. A simple discrete analogue of Proposition 1.1 was employed in [5, 6] and we give such a result here.*

Let $\{\mathrm{p}_\ell\}$ *be a sequence of positive values that satisfy* $\{\mathrm{p}_{n+1} - \mathrm{p}_n\} \leq -hA\mathrm{p}_n + hB \max_{n-m\leq\ell\leq n} \mathrm{p}_\ell$ *for* $n \in \{0, 1, 2, \cdots\}$ *(where* $h > 0$*). If* $0 < hA < 1$ *and* $0 \leq B \leq A$ *then* $\mathrm{p}_n \leq \widehat{\mathrm{p}}\nu^n$ *for* $n \in \{0, 1, 2, \cdots\}$, *where* $\widehat{\mathrm{p}} = \max\{\mathrm{p}_{-m}, \mathrm{p}_{1-m}, \cdots \mathrm{p}_0\}$ *and* $\widehat{\nu}$ *is the zero of largest modulus of the polynomial* $\nu^{m+1} - (1 - hA)\nu^m - hB$. *(Show that* $|\widehat{\nu}| \leq 1$. *The result follows by induction.)*

2. INTRODUCTION TO DDEs AND DISCRETE ANALOGUES

We recall DDEs and some related equations. See Hale's contribution in [1] for a pure mathematician's historical perspective and references to the theory; see Erneux [14] for applications; for numerics, and additional citations, see [2, 10]. The equation $y'(t) = f\big(t, y(t), y(t - \tau(t))\big)$ with non-trivial $\tau(t) \geq 0$ and continuous f is an example of a DDE. For our assumptions, see §2.1. We interpret y' in (2.1a) as the right-hand derivative y'_+, that is

$$y'_+(t) = f\big(t, y(t), y(t - \tau(t))\big) \quad (t \geq t_0). \tag{2.1a}$$

A solution $y(t)$ will exist and be specified by φ if one requires

$$y(t) = \varphi(t) \text{ for } t \in [t_*, t_0] \text{ where } t_* = \inf_{s\geq t_0} \{s - \tau(s)\}. \tag{2.1b}$$

An interval $[t_*, t]$ is interpreted as $(-\infty, t]$ if $\inf_{s\geq t_0} s - \tau(s) = -\infty$. We consider scalar equations, or systems with $\varphi(t)$ and $y(t) \in \mathbb{E}^m$ ($\mathbb{E} = \mathbb{R}$ or sometimes $\mathbb{C}$). We assume that φ is bounded and continuous on $[t_*, t_0]$, with finite norm $\|\varphi\|^{[t_*,t_0]}$ (for bounded ψ, we write $\|\psi\|^{[a,b]} = \sup_{a<s<b} |\psi(s)|$ where $|\cdot|$ is any norm in $\mathbb{R}^m$ and a or b may be infinite), and has bounded piecewise-continuous r-th order derivatives ($r \in \mathbb{Z}_+$) on $[t_*, t_0]$.

It is useful to denote the solution of (2.1a)–(2.1b) by

$$y(t) \equiv y(\varphi; t) \quad (t \geq t_0). \tag{2.1c}$$

Ex.2.1. *Consider the DDE* $y'(t) = -\alpha y(t) + \beta y(t - \tau^+)$ $(t \geq 0)$, *with* $\tau^+ > 0$. *By substitution,* $y(t) = C\exp(\zeta t)$ *is a solution of this equation for any* C *and any* ζ *such that* $\chi(\zeta) = 0$ *where* $\chi(\zeta)$ *is* $\zeta + \alpha - \beta\exp(-\tau^+\zeta)$, *the so-called* auxiliary function, *and a quasi-polynomial in* ζ. *In the notation of (1.1),* $\chi(\zeta) = -\mathcal{Q}(-\zeta)$. *If* C *equals* $k > 0$ *then* $y(t)$ *becomes* $k\exp(-\gamma t)$ *which is decreasing, so it satisfies* $y'(t) \leq -\alpha y(t) + \beta \max_{s\in[t-\tau^+,t]} y(s)$.

2.1. Assumptions for the DDE

In much of the literature related to (2.1), the '*delayed argument*' $t - \tau(t)$ in (2.1a) is taken as $t - \tau^+$, or as $t - \tau_\natural(t)$ where $\tau_\natural(t) \in [0, \tau^+]$, for $\tau^+ \in (0, \infty)$. In contrast, we permit $\tau(t)$ to be unbounded for $t \geq t_0$, provided that, with $t_* \leq t_0$, $t_* \leq t - \tau(t) \leq t$ for $t \geq t_0$ and $t - \tau(t) \to \infty$ as $t \to \infty$. The relaxed conditions on $\tau(\cdot)$, together with the possibility that $\varphi'(t_0) \neq f\big(t_0, \phi(t_0), \varphi(t_0 - \tau(t_0))\big)$, lead us to take the derivative in (2.1a) as the right-hand derivative, though (since $t - \tau(t) \to \infty$) this is ultimately (for $t \in [t^0, \infty)$, say) the two-sided derivative.

Assumption 2.1. *The problem (2.1) has a unique continuous solution.*

The last assumption is satisfied if f satisfies uniform Lipschitz conditions; these can be replaced by conditions involving, *e.g.*, one-sided Lipschitz conditions.

Remark 2.1. *Eqn. (2.1a) is an example of a* retarded functional differential equation *(RFDE; see [23, p.3]) such as* $y'(t) = \mathfrak{f}(t, y_t)$ *where* y_t *is a function defined by the restriction of* y *to* $[t_*, t]$. *Such equations are* Volterra *equations; indeed (2.1) yields* $y(t) = y(t_0)+\int_{t_0}^{t} f(s, y(s), y(s-\tau_\natural(s)))\mathrm{d}s$ $(t \geq t_0)$*, — in essence, using (2.1b), a* Volterra integral equation.

2.2. Stability of systems of delay differential equations

The *problem* (2.1) is called *contractive* [10] if (for any $\varphi_{1,2}$ for which $y_i(t) = y(\varphi_i; t)$ are defined for $t \geq t_0$) we have $\|y(\varphi_1; t) - y(\varphi_2; t)| \leq \sup_{s\in[t_*,t_0]} \|\varphi_1(s) - \varphi_2(s)\|$. The investigation of contractivity is similar to that of *stability of solutions with respect to ("w.r.t.") perturbations in* φ (stability *of a solution* w.r.t. initial conditions– for a formal definition see Definition A.1). Here, one seeks statements about $\delta y(t) := y(\varphi + \delta\varphi; t) - y(\varphi; t)$ for 'admissible' perturbations $\delta\phi$ and a specific φ (equivalently, a specific $y(\varphi; t)$). For the non-linear problem, *one solution* $y(\varphi; t)$ *may be stable while another solution* $y(\widehat{\varphi}; t)$ *may be unstable.* In the case of (2.1a)– (2.1b) we have

$$\delta y'(t) = f_\natural\big(t, \delta y(t), \delta y(t - \tau(t))\big) \quad (t \geq t_0), \quad \delta y(t) = \delta\varphi(t) \quad (t \in [t_*, t_0]), \tag{2.2}$$

$$f_\natural(t, u, v) := f(t, u + y(t), v + y(t - \tau(t)) - f(t, y(t), y(t - \tau(t))), \tag{2.3a}$$

and the problem to address returns to the boundedness or decay of the solution of a system ((2.3), with $f_\natural(t, 0, 0) \equiv 0$, instead of (2.1a)– (2.1b)). We can thus follow a common convention and discuss stability of "the zero solution": it is then assumed that $f(t, 0, 0) \equiv 0$ and $y(t) \equiv 0$ satisfies (2.1) if $\varphi(t) \equiv 0$. It seems denigrating to refer to the zero solution as the *trivial* solution; a change of dependent variable allows one to consider the stability of any solution; for further reading see [2, 10, 12, 18, 21, 20] *etc.*

2.3. Discrete relations

Modellers of problems with memory often employ discrete relations rather than functional equations. Typical of such discrete problems are those governed by equations of one of the following types (where $N(n) \in \mathbb{N}$ for $n \in \mathbb{Z}_+$):

$$U_{n+1} = F_0(n; U_{n+1}, U_n, U_{n-1}, \cdots, U_{n+1-N(n)}) \quad (n \in \mathbb{Z}_+); \tag{2.4a}$$

$$U_{n+1} = F_1(n; U_n, U_{n-1}, \cdots, U_{n+1-N(n)}) \quad (n \in \mathbb{Z}_+); \tag{2.4b}$$

$$U_{n+1} = U_n + F_2(n; U_n, U_{n-1}, \cdots, U_{n+1-N(n)}) \ (n \in \mathbb{Z}_+), \tag{2.4c}$$

with, in each case, appropriate conditions on $F_{0,1,2}$ and prescribed values of $\{U_{-\ell}\}$ (for appropriate $\ell \in \mathbb{Z}^+$) that define a solution. A problem (2.4a) can often be recast in the form (2.4b) or (2.4c). An important variant of (2.4c) is

$$U_{n+s} = U_n + F_3(n, s; U_n, U_{n-1+s_1}, \cdots, U_{n+1-N(n)+s_N}). \tag{2.4d}$$

(for $n \in \mathbb{Z}_+$; $s \in (0, 1], s_1, \cdots, s_N \in [0, 1)$). Problems of the form (2.4c), (2.4d) arise in numerical methods for (2.1); standard k-step relations used in the numerics of ODEs are also examples. Corresponding *discrete*, and '*discrete-continuous*', variants of Halanay's lemma will therefore be of interest.

Ex.2.2. *Consider the numerical solution of* $y'(t) = f(t, y(t), y(t - \tau^+))$ $(t \geq t_0)$ *where* $\tau^+ > 0$, *with* $y(t) = \varphi(t)$, $t \in [t_0 - \tau^+, t_0]$. *For* $\Theta \in [0, 1]$, $h = \tau^+/N$ $(N \in \mathbb{N})$, $t_n = t_0 + nh$ $(-N \leq n \in \mathbb{Z})$, *we seek* $\widetilde{y}_n \equiv \widetilde{y}(t_n) \approx y(t_n)$ *that satisfy*

$$\widetilde{y}_{n+1} - \widetilde{y}_n = h\{(1-\Theta)f(t_n, \widetilde{y}_n, \widetilde{y}_{n-N}) + \Theta f(t_{n+1}, \widetilde{y}_{n+1}, \widetilde{y}_{n+1-N})\ \ (n \in \mathbb{Z}_+); \tag{2.5a}$$

$$\widetilde{y}_\ell = \varphi(t_\ell) \quad \ell \in \{-N, 1-N, \cdots, 0\}; \tag{2.5b}$$

cf.*(2.4). (If* $\Theta \neq 0$, *the equations for* $\widetilde{y}_{n+1}$ *are implicit, and conditions are required to ensure existence and uniqueness.) Our example is special; for general DDEs and stepsizes the numerical solution is usually densely defined on* $[t_0, \infty)$.

At issue, above, is that (2.5) involves differences rather than derivatives. However, discrete recurrence relations can be associated with right- differentiable densely-defined functions. For example, consider (2.4c) (assumed to hold for $n \in \mathbb{Z}_+$ with prescribed initial values $\{U_{-\ell}\}_{\ell \in \mathbb{Z}_+}$). Pick a monotonic increasing $\{t_\ell^\dagger\}_{\ell \in \mathbb{Z}}$ (the choice $\cdots < t_{-2}^\dagger < t_{-1}^\dagger < t_0^\dagger < t_1^\dagger < t_3^\dagger < \cdots$ has a scaling effect, so in any given case restrictions will be applied). Assuming (2.4c), write

$$\widehat{U}(t_{n+s}) = U_n + sF_2(n; U_n, U_{n-1}, \cdots, U_{n+1-N(n)}) \quad (s \in [0, 1),\ n \in \mathbb{Z}_+). \tag{2.6}$$

If $\widehat{U}(t_{-\ell}) = U_{-\ell}$ for $\ell \in \mathbb{Z}_+$ then $\widehat{U}(t_n) = U_n$ for all $n \in \mathbb{Z}$; $\widehat{U}(\cdot)$ is the *simplest continuous dense extension* of $\{U_n\}$ on $\{t_n\}$. For $s \in [0, 1)$, $n \in \mathbb{Z}_+$,

$$\widehat{U}'_+(t_{n+s}) = \frac{1}{t_{n+1} - t_n} F_2(n; \widehat{U}(t_n), \widehat{U}(t_{n-1}), \cdots, \widehat{U}(t_{n+1-N(n)})). \tag{2.7}$$

Thus, discrete recurrences (2.4) can be analyzed, via the form (2.4c), using densely defined extensions, and theory for (2.6) or (2.7). On the other hand, DDEs and RFDEs can sometimes be analyzed by discrete recurrences (2.4).

2.4. Itô stochastic DDEs

Our discussion of (2.1) can be extended to Itô stochastic DDEs (*SDDEs*)

$$\mathrm{d}Y(t) = F(t, Y(t), Y(t - \tau^+))\,\mathrm{d}t + G(t, Y(t), Y(t - \tau^+))\,\mathrm{d}W(t)\ (t \geq 0) \tag{2.8}$$

where $W(t)$ is a one-dimensional standard Wiener process. With $h = \tau^+/N$, an extension of Example 2.2 gives approximations $\widetilde{Y}_n \equiv \widetilde{Y}_n(\Phi) \approx Y(nh)$ satisfying

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n + (1-\theta)hF_n + \theta hF_{n+1} + \sqrt{h}G_n\xi_n \quad (\xi_n \in \mathcal{N}(0, 1)), \tag{2.9}$$

$$F_r = F(rh, \widetilde{Y}_r, \widetilde{Y}_{r-N}),\ G_r = G(rh, \widetilde{Y}_r, \widetilde{Y}_{r-N});\ \widetilde{Y}_{-\ell} = \Phi(t_{-\ell})\ (\ell \in \{0, 1, \cdots, N\}).$$

See [4, 5, 6]; the necessary theoretical detail is referred to therein. Of interest at present is the fact that a trajectory $Y(t)$ is continuous but not differentiable (indeed, (2.8) is better expressed in integral equation form). The issue here is that Dini derivatives play an essential role in applications of Halanay-type theory to inequalities involving expectations $\mathsf{E}(Y(t)^2)$.

2.5. **Numerical solutions and their properties**

In the *elementary* discussion of the numerical solution of ODEs, it is common to define an approximate solution on a mesh $\{t_0 + nh\}_{n\in\mathbb{Z}}$ (where $h > 0$). For a more realistic discussion, a non-uniform mesh is constructed in adaptive fashion, and a densely-defined extension of the approximation is introduced (the approximation is defined for, say, $t \in [t_0, T]$). It is natural, in the numerical solution of DDEs, to have an underlying mesh $\mathcal{T} := \{t_0, t_1, t_2, \ldots\}$ but to *require* (in general) a densely-defined approximation; we need this to access values of the approximation with a lagging argument when $t_\ell - \tau(t_\ell) \notin \mathcal{T}$.

2.6. **General Θ-methods for DDEs**

Though simple, the Θ-methods provide a basis for a discussion of approximating versions of (2.1). We discuss two types of Θ-methods: *continuous one-leg* Θ*-methods* and *continuous linear* Θ*-methods* for the DDE (2.1). For related ODE methods, see, *e.g.*, [16]. Our methods are associated with a Θ-formula, described in this section, and a variable-stepsize grid

$$\mathcal{T} := \{t_0 < t_1 < t_2 < \ldots\}, \text{ where } t_{n+s} = \{1-s\}t_n + st_{n+1} \text{ for } s \in [0,1] \tag{2.10}$$

(or $t_{n+s} = t_n + sh_n$) and $t_n \to T$ as $n \to \infty$. For an arbitrary function ψ we write $\psi_{n+s} = \psi(t_{n+s})$; thus, $\tau_{n+s} = \tau(t_n + sh_n)$. We suppose

$$0 < h_\star \le h_n \le h^\star < \infty, \tag{2.11}$$

$$t_{n+s} - \tau_{n+s} \notin (t_n, t_{n+s}] \text{ for } s \in [0,1]. \tag{2.12}$$

We rely on (2.12), which holds if $h^\star \le \inf_{t\ge t_0}\{\tau(t)\}$, to avoid a discussion on how to define the methods when (2.12) is violated. If it exists, $\widetilde{y}(t)$ $(t \ge t_0)$ depends on $\Theta \in [0,1]$ and on the type (1-leg or linear) of method. To be precise,

$$\widetilde{y}(t) \equiv \widetilde{y}^\sharp(t), \quad \widetilde{y}^\sharp(t) \equiv \widetilde{y}^\sharp_\Theta(\phi, \mathcal{T}; t),\ \Theta \in [0,1] (t \ge t_0), \tag{2.13a}$$

where $\sharp$ is an indicator of the type (1-leg or linear) of the method, as in

$$\widetilde{y}^\sharp(t) = \widetilde{y}^{\textit{1-leg}}(t) \text{ or } \widetilde{y}^\sharp(t) = \widetilde{y}^{\textit{lin}}(t). \tag{2.13b}$$

Thus, $\widetilde{y}^{\textit{1-leg}}_{1/2}(\phi, \mathcal{T}; t)$ is a continuous mid-point approximation; $\widetilde{y}^{\textit{lin}}_0$, $\widetilde{y}^{\textit{lin}}_{1/2}$, $\widetilde{y}^{\textit{lin}}_1(\phi, \mathcal{T}; t)$ are continuous Euler, trapezium, or backward Euler approximations. For a chosen approximation $\widetilde{y}^\sharp(\cdot)$, let

$$\widetilde{f}^\sharp_{n+s} \quad = f\big(t_{n+s}, \widetilde{y}^\sharp(t_{n+s}), \widetilde{y}^\sharp(t - \tau(t_{n+s}))\big). \tag{2.14}$$

Now. the continuous one-leg and linear Θ-methods are defined by

$$\widetilde{y}^{\textit{1-leg}}(t_n + sh_n) = \widetilde{y}^{\textit{1-leg}}(t_n) + sh_n \widetilde{f}^{\textit{1-leg}}_{n+\Theta}, \quad s \in [0,1], n \in \mathbb{N}, \tag{2.15}$$

$$\tilde{y}^{\textit{lin}}(t_n + sh_n) = \tilde{y}^{\textit{lin}}(t_n) + sh_n\big[(1-\Theta)\tilde{f}^{\textit{lin}}_n + \Theta \tilde{f}^{\textit{lin}}_{n+1}\big], \quad s \in [0,1], n \in \mathbb{N}. \tag{2.16}$$

Remark 2.2. *If they exist, both (2.15) and (2.16) are* densely defined*, for all* $t \ge t_0$*, and are piecewise-linear (different extensions are possible). For general* $\tau(\cdot)$*, there exist integer* n*, and* $\mu \in [0,1]$ *such that* $t_{n+\mu} - \tau(t_{n+\mu}) \notin \mathcal{T}$*. If* $t_{n+\mu} - \tau(t_{n+\mu}) \in (t_k, t_{k+1})$ *(by assumption,* $k < n$*) then* $\widetilde{f}^\sharp_{n+\mu}$ *depends on* $\widetilde{y}^\sharp(t_{n+\mu} - \tau(t_{n+\mu}))$ *which can be evaluated using* $\widetilde{y}^\sharp(t_k)$*, and* $\widetilde{f}^{leg}_{k+\mu}$ *or* $\widetilde{f}^{\textit{lin}}_k$*,* $\widetilde{f}^{\textit{lin}}_{k+1}$*.*

Assumption 2.2. *Assume the existence of $\delta' < 0$ and $\delta'' > 0$ such that when $\delta \in [\delta', \delta'']$ there exists a function $\check{F}$ with the property that for any $t \in [t_0, T]$ the equation $x = y + (\delta)f(t, x, w)$ has a unique solution x satisfying*

$$x = \check{F}(t, \delta; y, w). \tag{2.17}$$

Ex.2.3. *Suppose that $f(t, x, w) = -\alpha(t)x + \beta(t)w$; then, provided $1 + \delta \cdot \alpha(t) \neq 0$, $\check{F}(t, \delta; y, w) = \{y + \delta \cdot \beta(t)w\}/\{1 + \delta \cdot \alpha(t)\}$.*

Subject to reasonable assumptions, the existence of F can be established by fixed point theory. Results verifying Assumption 2.2 can be borrowed from corresponding theory of the numerics of ODEs. For example, we have: *Suppose that $\|f(t, x', w) - f(t, x'', w)\| \leq \xi(t, w) \|x' - x''\|$ for $t \geq t_0$, and $w \in \mathbb{R}^m$. Then F exists when $|\delta t| \sup_{t,w}\{\xi(t, w)\} < 1$.* Results based on one-sided Lipschitz conditions or a lub logarithmic Lipschitz constant (see (3.11)) arise on generalizing a result for ODEs – see [17, p.331], and [29].

Theorem 2.1. *Given Assumption 2.2, $\widetilde{y}^{\text{1-leg}}$ and $\widetilde{y}^{\text{lin}}$ exist and are unique if h^* is sufficiently small.*

Example 2.2 illustrated an exception to the need for a densely-defined extension; further exceptions arise in the case $\tau(t) \equiv \tau^+$ with the linear Θ-methods $(0 \leq \Theta \leq 1)$ and the one-leg Θ-methods if $\Theta = 0$ or if $\Theta = 1$. A dense extension *is* required for the one-leg methods if $\Theta \in (0, 1)$. Even if a dense extension is not required it can be supplied as an extra, and for the piecewise-linear extensions adopted here, the extension has right-hand derivatives. Results for RFDEs and DDEs can then be applied (at least in principle) to Θ-methods:

Remark 2.3. *Suppose $\check{F}$ satisfies (2.17) with $\delta' = \inf_n(\Theta - 1)h_n$ and $\delta'' = \sup_n \Theta h_n$. Then there exist one-leg functions $\widehat{f}^{\text{1-leg}}(t, u, v)$ and $\tau^{\text{1-leg}}(t)$ such that the one-leg approximations satisfy equations of the form $(\frac{\mathrm{d}}{\mathrm{d}t})_+ \widetilde{y}^{\text{1-leg}}(t) = \widehat{f}^{\text{1-leg}}(t, \widetilde{y}^{\text{1-leg}}(t), \widetilde{y}^{\text{1-leg}}(t - \tau^{\text{1-leg}}(t)))$. We can show that, for $t \in [t_n, t_{n+1})$, $(\dfrac{\mathrm{d}}{\mathrm{d}t})_+ y^{\text{1-leg}}(t) = f\Big(t, F\big(t_{n+\Theta}, (t_{n+\Theta} - t); \widetilde{y}^{\text{1-leg}}(t), \widetilde{y}^{\text{1-leg}}(t - \tau(t_{n+\Theta}))\big), \widetilde{y}^{\text{1-leg}}(t - \tau(t_{n+\Theta}))\Big)$. Expressions for $\widehat{f}^{\text{1-leg}}$ are intricate as they involve Θ and $\check{F}$ in Assumption 2.2.*

Remark 2.4. *For linear methods, we can compute $\widetilde{y}^{\text{lin}}(t_{n+1})$ by using F to solve*

$$\begin{aligned} \widetilde{y}^{\text{lin}}(t_{n+1}) = \big\{\tilde{u}^{\text{lin}}(t_n) + h_n(1 - \Theta)f(t_n, \widetilde{y}^{\text{lin}}(t_n), \widetilde{y}^{\text{lin}}(t - \tau(t_n))\big\} + \\ + \Theta h_n f(t_{n+1}, \widetilde{y}^{\text{lin}}(t_{n+1}), \widetilde{y}^{\text{lin}}(t - \tau(t_{n+1})). \end{aligned} \tag{2.18}$$

The resulting value of $\widetilde{y}^{\text{lin}}(t_{n+1})$ can be used to obtain $(\frac{\mathrm{d}}{\mathrm{d}t})_+ \widetilde{y}^{\text{lin}}(t)$ on $[t_n, t_{n+1})$ in terms of $\widetilde{y}^{\text{lin}}(t_n)$, $\widetilde{y}^{\text{lin}}(t_n - \tau(t_n))$, and $\widetilde{y}^{\text{lin}}(t_{n+1} - \tau(t_{n+1}))$.

In general, one may establish that a numerical approximation can satisfy a RFDE but its form will be complicated. It is usually simpler to use "discrete-continuous" recurrence relations like that found in the following example.

Ex.2.4. *Suppose $f(t, u, v) = -\alpha(t)u + \beta(t)v$. Let $\alpha(t) \geq 0$. If $1 + \Theta h_n \alpha(t_{n+\Theta}) \neq 0$ for $n \in \mathbb{N}$ then $\widetilde{y}^{\text{1-leg}}$ exists. Abbreviating $\widetilde{y}^{\text{1-leg}}(t_{n+s})$ as $\widetilde{y}_{n+s}$, $\alpha(t_{n+\Theta})$ as $\alpha_{n+\Theta}$,$\beta(t_{n+\Theta})$ as $\beta_{n+\Theta}$, and $\tau(t_{n+\Theta})$ as $\tau_{n+\Theta}$, we obtain, for $s \in [0, 1]$,*

$$\widetilde{y}_{n+s} = \left\{1 - \frac{sh_n\alpha_{n+\Theta}}{1 + \Theta h_n\alpha_{n+\Theta}}\right\}\widetilde{y}_n + sh_n\frac{\beta_{n+\Theta}}{1 + \Theta h_n\alpha_{n+\Theta}}\widetilde{y}(t_{n+\Theta} - \tau_{n+\Theta}). \tag{2.19}$$

An expression for $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_+ \widetilde{y}^{\text{1-leg}}(t)$ follows. Significantly, if we write $t_{n+\Theta} - \tau_{n+\Theta}$ as $t_{k+s'}$, $\widetilde{y}(t_{n+\Theta} - \tau_{n+\Theta})$ is given by replacing n, s by k, s' on (2.19).

3. SOME HALANAY-TYPE THEORY

The starting point for application of Halanay-type theory is an inequality of suitable type involving some kind of derivative, or (as we see) a difference.

Suppose y is a differentiable scalar-valued function defined on $[t_0, \infty)$, and suppose that $\mathrm{p}(t) = |y(t)|$. Where $y(t)$ changes sign, the conventional derivative of $\mathrm{p}(t)$ fails to exist although $\mathrm{p}'_+(t)$ does exist (this difficulty disappears if one picks $\mathrm{p}(t) = |y(t)|^2$). But extensions of Halanay's Lemma that hold for right-hand derivatives or (as noted in §2.4) for Dini derivatives will be of interest.

3.1. General remarks

With appropriate conditions on f, a nonlinear inequality

$$\mathrm{p}'_+(t) \le f\big(t, \mathrm{p}(t), \sup_{s\in[t-\tau_\natural(t),t]} \mathrm{p}(s)\big), \tag{3.1}$$

is associated with some relation $\mathrm{p}'_+(t) = f\big(t, \mathrm{p}(t), \sup_{s\in[t-\tau_\natural(t),t]} \mathrm{p}(s)\big) + \varrho(t)$ $(t \ge t_0)$ subject to the constraint $\varrho(t) \ge 0$ $(t \ge t_0)$ – and likewise with a Dini derivative. For a given ϱ a solution, if such exists, is determined (though possibly not uniquely) by a condition of the type

$$\sup_{s\in[t_*,t_0]} \mathrm{p}(s) = \widehat{\mathrm{p}}, \text{ where } t_* = \inf_{t\ge t_0}\{t - \tau(t)\}. \tag{3.2}$$

We now introduce a notation that simplifies the writing of inequalities of the type (3.1). For a given delayed argument $t - \tau(t)$, a function corresponding to a bounded function ψ, denoted $\overleftarrow{\psi}$, is defined by

$$\overleftarrow{\psi}(t) := \sup_{t-\tau(t)\le s\le t} \psi(s). \tag{3.3}$$

Thus (3.1) can be written $\mathrm{p}'(t) \le f\big(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t)\big)$. Note that, if ψ is continuous, there exists some function $\widehat{\tau}(\cdot)$ such that $\overleftarrow{\psi}(t) = \psi(t - \widehat{\tau}(t))$ with $t - \widehat{\tau}(t) \in [t - \tau(t), t]$; thus $t - \widehat{\tau}(t) \to \infty$ as $t \to \infty$ and $\limsup_{t\to\infty} \psi(t) = \limsup_{t\to\infty} \overleftarrow{\psi}(t)$.

In applications to stability, we often substitute a Lyapunov function in place of p. The notation frequently used for Lyapunov-type functions is V; see §2.2. Boundedness (or decay) of $V(t; y(t))$ implies boundedness (resp. decay) of $y(t)$.

3.2. Some linear inequalities associated with DDEs

Now consider the system of linear DDEs with multiple lags

$$y'(t) = A(t)y(t) + \sum_{\ell=0}^{L} B_\ell(t)y(t - \tau_\ell(t)), \text{ with } \widehat{\tau}(t) := \max_\ell \tau_\ell(t), \tag{3.4}$$

$(t \in [t_0, \infty))$ where $y \in ([t_0, \infty) \to \mathbb{R}^n)$, $A, B_\ell \in C([t_0, \infty) \to \mathbb{R}^{n\times n})$, $\tau_\ell \in C[t_0, \infty)$ $(\ell \in \{0, \cdots, L\})$, and $t - \widehat{\tau}(t) \to \infty$ as $t \to \infty$. We now obtain a scalar inequality for the norm of a solution $y(\varphi; t)$, defined by φ.

We denote ([29]; *cf.* Definition A.5) the *measure* or *logarithmic norm* of A by

$$\mu_{\|\cdot\|}(A) := \mathrm{D}^+\|I + \xi A\|\Big|_{\xi=0} = \lim_{\delta\xi\to 0} \frac{\|I + |\delta\xi| \times A\| - 1}{|\delta\xi|} \in (-\infty, \infty). \tag{3.5}$$

Expressions for $\mu_{\|\cdot\|_1}(A)$, $\mu_{\|\cdot\|_2}(A)$, $\mu_{\|\cdot\|_\infty}(A)$ are in the literature. A can depend on t ($A = A(t)$, say). For a scalar, $\mu_{\|\cdot\|}\big(a(t)\big) = a(t)$.

Lemma 3.1. *Suppose* $y \in ([t_*, \infty) \to \mathbb{R}^n)$ *satisfies (3.4). Then* $(\frac{\mathrm{d}}{\mathrm{d}t})_+ \|y(t)\| \le \mu_{\|\cdot\|}(A(t))\ \|y(t)\| + \sum_{\ell=0}^{L} \|B_\ell(t)\|\ \|y(t - \tau_\ell(t))\|$. *Further, then for* $t \in [t_0, \infty)$,

$$(\frac{\mathrm{d}}{\mathrm{d}t})_+ \|y(t)\| \le \Big\{\mu_{\|\cdot\|}\big(A(t)\big)\Big\}\ \|y(t)\| \ +\ \sum_{\ell=0}^{L} \|B_\ell(t)\| \sup_{s\in[t-\widehat{\tau}(t),t]} \|y(s)\|. \tag{3.6}$$

Proof.

$$(\frac{\mathrm{d}}{\mathrm{d}t})_+ \|y(t)\| = \lim_{\delta t\to 0+} \frac{\|y(t+\delta t)\| - \|y(t)\|}{\delta t} = \lim_{\delta t\to 0+} \frac{\|y(t) + \delta t\ y'(t)\| - \|y(t)\|}{\delta t} \tag{3.7}$$

$$= \lim_{\delta t\to 0+} \frac{\|[I + \delta t\ A(t)]y(t) + \delta t\ \sum_{\ell=0}^{L} B_\ell(t) y(t - \tau_\ell(t))\| - \|y(t)\|}{\delta t}$$

$$\le \lim\nolimits_{\delta t\to 0+} \frac{\|I + \delta t\ A(t)\| - 1}{\delta t} \|y(t)\| + \sum_{\ell=0}^{L} \|B_\ell(t)\|\ \|y(t - \tau_\ell(t))\|.$$

The remainder of the proof is now straightforward. ■

Similar results have been stated for fixed lags and in terms of the spectral abscissa of $\frac{1}{2}(A + A^*)$ which is $\mu_{\|\cdot\|_2}(A)$. If $A \in \mathbb{R}^{n\times n}$, the conditions $\mu_{\|\cdot\|_1}(A) < 0$, $\mu_{\|\cdot\|_\infty}(A) < 0$ amount to requiring A has negative diagonal elements and is diagonally dominant (respectively, by rows or by columns).

Ex.3.1. *If we seek effective results, we require* $A(t) \ne 0$ *in (3.4). Consider the* pure delay *equation* $y'(t) = \lambda(t) y(t - \tau(t))$, *which fails this requirement. As* $y(t - \tau(t)) = y(t) - \tau(t) y'(t - s(t)\tau(t))$ *(for some* $0 < s(t) < 1$*), we obtain*

$$y'(t) = \lambda(t) y(t) - \{(\lambda(t))^2 \tau(t)\} \times y(t - \tau^\diamond(t)), \tag{3.8}$$

with $\tau^\diamond(t) := [1 + s(t)]\tau(t)$, *and this is of the form (3.4) with non-trivial* $A(t)$.

3.3. Inequalities for systems of nonlinear DDEs

Now consider the nonlinear system of DDEs

$$y'_+(t) = f(t, y(t), y(t - \tau(t))) \quad (t \ge t_0) \tag{3.9}$$

with initial condition $y(t) = \varphi(t)$. In the case of stability of the zero solution, we assume $f(t, 0, 0) = 0$. There are various ways to proceed in discussing qualitative behaviour and in obtaining inequalities. For problems that can be addressed by the method of steps, one may be able to obtain global results by considering problems of the form $y'_n(t) = f(t, y_n(t), y_{n-1}(t - \tau(t)))$ on a sequence of subintervals $[t^n, t^{n+1}]$. However, an approach similar to the viewpoint of Halanay involves writing (3.9) as a perturbation of the differential equation $y'(t) = f(t, y(t), y(t))$, in the form

$$y'(t) = f(t, y(t), y(t)) + f_\sharp(t, y(t), y(t - \tau(t))), \tag{3.10a}$$

$$f_\sharp(t, u_1, u_2) := \{f(t, u_1, u_2) - f(t, u_1, u_1)\}. \tag{3.10b}$$

Ex.3.2. *In the scalar case we can obtain inequalities from (3.10), for example by first multiplying by sign*$\{y(t)\}$ *to bound* $(\frac{\mathrm{d}}{\mathrm{d}t})_+ |y(t)|$ *or alternatively by multiplying by* $2y(t)$ *to bound* $(\frac{\mathrm{d}}{\mathrm{d}t})\{[y(t)]^2\}$. *Terms involving* $f_\sharp(t, y(t), y(t - \tau(t)))$ *can be bounded using the sensitivity of* $f(t, u_1, u_2)$ *to* u_2.

Definition 3.1. *Let $\mathfrak{f} \in D \to \mathbb{E}^m$, where $D \subset \mathbb{E}^m$ is a path-connected set, and let $\mathfrak{f}$ have Lipschitz constant $L_{\|\,\|}[\mathfrak{f}] = \sup_{u_{1,2}\in D} \dfrac{\|\mathfrak{f}(u_1) - \mathfrak{f}(u_2)\|}{\|u_1 - u_2\|}$ over D. Let $I + \delta\cdot\mathfrak{f}$ be the map $x \to x + \delta\times\mathfrak{f}(x)$ ($\delta \in \mathbb{E}$). Then the* (lub) logarithmic Lipschitz constant of $\mathfrak{f}$ over D *is*

$$M_{\|\,\|}[\mathfrak{f}] = \lim_{\delta\to 0+} \frac{L_{\|\,\|}[I + \delta\cdot\mathfrak{f}] - 1}{\delta}. \tag{3.11}$$

Here, $\mathfrak{f}$ can have a parameter $t \geq t_0$: $[\mathfrak{f}(t)](x) = \mathfrak{F}(t;x)$, say.

Definition 3.1 generalizes the concept of the logarithmic norm (3.5); in it, we do not require differentiability of $\mathfrak{f}$. In lieu of (3.4), suppose (with $y(t) \in \mathbb{R}^m$) that

$$y'(t) = f\big(t, y(t, y(t - \tau_1(t), \cdots, y(t - \tau_L(t))))\big) \quad (t \geq t_0) \tag{3.12}$$

with the usual conditions on $\{\tau_\ell(\cdot)\}$ and assume that $f(t, 0, 0, \cdots, 0) \equiv 0$ so that the zero solution is an equilibrium of (3.12). We write

$$\begin{aligned} F(t,u) &= f(t,u,u,\cdots,u), && (3.13\text{a})\\ G(t,u,u_1,\cdots,u_L) &= f(t,u,u_1,\cdots,u_L) - f(t,u,u,\cdots,u). && (3.13\text{b}) \end{aligned}$$

(Here, $F(t,0) = G(t,0,\cdots,0) \equiv 0$.) Clearly (3.12) implies

$$y'(t) = F\big(t, y(t)\big) + G\big(t, y(t), y(t-\tau_1(t)), \cdots, y(t - \tau_L(t))\big)\big) \quad (t \geq t_0) \tag{3.14}$$

which displays the equation as a perturbation of $y'(t) = F(t, y(t))$. In the case that $F(t,u) \equiv 0$, we can substitute $y(t - \tau_\ell(t)) = y(t) - \int_{t-\tau_\ell(t)}^{t} y'(s)\mathrm{d}s = y(t) - y'(t - \eta\tau_\ell(t))$ and use (3.12) to obtain a new expression.

From (3.14) we seek an inequality for a Lyapunov function(al). For example, we can take $V(t, u(t)) = \|u(t)\|^2$ and seek an inequality by taking the inner-product of (3.14) with $y(t)$. Here, we develop a bound for $\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)_+\|y(t)\|$ where $\|\cdot\|$ is any norm. Eqn (3.7) holds and now yields

$$(\frac{\mathrm{d}}{\mathrm{d}t})_+\|y(t)\| \leq M_{\|\,\|}[F(t,\cdot)] + \|G\big(t, y(t), y(t-\tau_1(t), \cdots, y(t-\tau_L(t)))\big)\|. \tag{3.15}$$

Lemma 3.2 (below) follows if we assume Lipschitz conditions for $f(t, u_0, u_1, \cdots, u_L)$ with respect to $\{u_\ell\}$, and, following from this assumption, suppose that

$$\|G\big(t, u_0, u_1(t), \cdots, u_L)\big)\| \leq \sum_{\ell=0}^{L} |b_\ell(t)|\,\|u_\ell\|. \tag{3.16}$$

Lemma 3.2. *Suppose that $y(t)$ satisfies (3.12), $f(t, 0, \cdots, 0) \equiv 0$, and (3.16) is valid. Then, with the notation in (3.11), (3.13) and (3.16), $(\frac{\mathrm{d}}{\mathrm{d}t})_+\|y(t)\| \leq M_{\|\,\|}[F(t,\cdot)]\|y(t)\| + \sum_{\ell=0}^{L}|b_\ell(t)|\,\|y(t - \tau_\ell(t))\|$ and*

$$(\frac{\mathrm{d}}{\mathrm{d}t})_+\|y(t)\| \quad \leq \quad M_{\|\,\|}[F(t,\cdot)]\,\|y(t)\| + \sum_{\ell=0}^{L}|b_\ell(t)| \sup_{s\in[t_*,t]}\|y(s)\|. \tag{3.17}$$

F is defined by (3.13). The inequality (3.17) is essentially linear. Nonlinear inequalities can often be obtained using (3.14); *cf.* Ex. 3.1. *Analogous results hold [3] for Θ-method approximations.*

Inequalities (3.6), (3.15), and (3.17) are candidates for the application of Halanay-type results, e.g., in the study of stability, asymptotic stability or exponential stability and of contractivity [2, 10, 18, 21].

3.4. **Variable-coefficient generalizations of Halanay's lemma**

We provide some generalizations and applications of results that may be regarded as extensions of Halanay's work. The following refinement of a result of Baker & Tang generalizes Proposition 1.1. (We state some results without proof.)

Theorem 3.3. *Suppose that*

$$\alpha(t) \geq \alpha_* > 0,\ \beta(t) \geq \beta_* \geq 0,\ \textit{for } t \in [t_0, \infty)\ , \tag{3.18a}$$

where α, β *are bounded and continuous on* $[t_0, \infty)$*,*

$$t - \tau(t) \leq t,\ t_* = \inf_{t\in[t_0,\infty)} t - \tau(t) \ \textit{and}\ t - \tau(t) \to \infty \ \textit{as}\ t \to \infty \tag{3.18b}$$

and p *is non-negative, bounded and continuous on* $[t_*, \infty)$*. If* $\mathrm{p}(t)$ *satisfies*

$$\mathrm{D}^+\mathrm{p}(t) \leq -\alpha(t)\mathrm{p}(t) + \beta(t) \sup_{t-\tau(t)\leq s\leq t} \mathrm{p}(s)\ (t \geq t_0), \tag{3.19}$$

and if there exists a value $\varsigma > 0$ *such that*

$$-\alpha(t) + \beta(t) \leq -\varsigma < 0 \qquad \textit{for} \quad t \geq t_0, \tag{3.20}$$

it follows that

$$\mathrm{p}(t) \leq ||\mathrm{p}||^{[t_*,t_0]}\ (t \geq t_0) \quad \textit{and} \tag{3.21a}$$

$$\mathrm{p}(t) \to 0 \quad \textit{as} \quad t \to \infty. \tag{3.21b}$$

Ex.3.3. *With* $y \in C([t_*, \infty) \to \mathbb{R})$*, let* $y(t) \equiv y(\phi; t)$ *satisfy* $y'(t) = a(t)y(t)+b(t)y(t-\tau(t))$ *for* $t \geq t_0$*, while* $y(t) = \phi(t)$ *for* $t \leq t_0$*. Suppose* $a(t)$ *and* $b(t)$ *are bounded and satisfy* $a(t)+|b(t)| \leq -\sigma < 0$*, for all* $t \geq t_0$ *(by assumption,* $a(t) < 0$*). Then* $|y(\phi; t)| \to 0$ *as* $t \to \infty$*. To see this, introduce* $\mathrm{p}(t) = |y(t)|$*. Then* $\mathrm{p}(t) = |\phi(t)|$*, if* $t \leq t_0$ *and, for* $t \geq t_0$*,* $\mathrm{p}'_+(t) = y'(t)\mathrm{sign}\{y(t)\}$*,* $= y'(t)\mathrm{sign}\{y(t)\}$ $= a(t)|y(t)|+b(t)y(t-\tau(t))\mathrm{sign}\{y(t)\}$*. Thus,* $\mathrm{p}'_+(t) \leq a(t)|y(t)|+|b(t)| \sup_{t-\tau(t)\leq s\leq t} |y(s)|$ *(*$t \geq t_0$*). On taking* $\alpha(t) = -a(t) > 0$ *and* $\beta(t) = |b(t)|$ *in Theorem 3.3, it follows that* $|y(t)|$ *is bounded and tends to zero as* $t \to \infty$*.*

3.5. **Refinements**

The *exponential* decay of $\mathrm{p}(t)$ as $t \to \infty$ (implied by (1.1)) is stronger than the simple decay in (3.21b). We seek a more informative result. We retain the preceding assumptions, and since $t - \tau(t) \to \infty$ as $t \to \infty$ there exists $t^0 \geq t_0$ such that $t - \tau(t) \geq t_0$ for all $t \geq t^0$. Note that (3.21a) implies that $\|\mathrm{p}(t)\|^{[t_0,t^0]} \leq \|\mathrm{p}\|^{[t_*,t_0]} < \infty$. Now suppose there exists a positive function $\sigma \in C[t_0, \infty)$ such that, for $t \geq t^0$,

$$0 < \sigma(t) \leq \alpha(t) - \beta(t) \exp\big(-\int_{t-\tau(t)}^{t} \sigma(s)\mathrm{d}s\big). \tag{3.22}$$

With that assumption, define the positive function u with

$$u(t) := \|\mathrm{p}\|^{[t_0,t^0]} \quad \text{for } t \in [t_0, t^0), \tag{3.23a}$$

$$u(t) := \|\mathrm{p}\|^{[t_0,t^0]} \exp\big(-\int_{t^0}^{t} \sigma(s)\mathrm{d}s\big) \quad \text{for } t \in [t^0, \infty). \tag{3.23b}$$

By definition, $\mathrm{p}(t) \leq u(t)$ for $t \in [t_0, t^0]$. As $u(t)$ is monotonic decreasing for $t \geq t^0$, $u(t-\tau(t)) = \overleftarrow{u}(t)\ (= \sup\limits_{t-\tau(t)\leq s\leq t} u(s))$ when $t \geq t^0$, and

$$u(t) = u(t-\tau(t)) \times \exp\big(-\int_{t-\tau(t)}^{t} \sigma(s)\mathrm{d}s\big), \tag{3.24}$$

$$u'(t) = -\sigma(t)u(t), \tag{3.25}$$

for $t \geq t^0$. We obtain $0 > -\sigma(t)u(t) \geq -\alpha(t)u(t) + \beta(t)u(t)\exp\big(-\int_{t-\tau(t)}^{t}\sigma(s)\mathrm{d}s\big)$ on multiplying (3.22) by $-u(t)$, so (3.24) and (3.25) yield

$$u'(t) \geq -\alpha(t)u(t) + \beta(t) \sup_{s\in[t-\tau(t),t]} u(s) \quad \text{for } t \geq t^0 \tag{3.26a}$$

while, by assumption,

$$\mathrm{D}^+\mathrm{p}(t) \leq -\alpha(t)\mathrm{p}(t) + \beta(t) \sup_{s\in[t-\tau(t),t]} \mathrm{p}(s) \quad (\text{for } t \geq t^0), \tag{3.26b}$$

$$\mathrm{p}(t) \leq u(t) \text{ for } t \in [t_0, t^0]. \tag{3.26c}$$

Theorem 3.4 extends a result given in [19] for the case where $\tau(\cdot)$ is bounded. To facilitate its proof, we define $u_k(t) = ku(t)$ (for $t \in [t_*, \infty)$) for arbitrary $k > 1$, so $u_k(t) = k\|\mathrm{p}\|^{[t_0,t^0]}$ for $t \in [t_0, t^0)$, and

$$u_k'(t) \geq -\alpha(t)u_k(t) + \beta(t) \sup_{s\in[t-\tau(t),t]} u_k(s) \quad \text{for } t \geq t^0. \tag{3.26d}$$

Theorem 3.4. *With the assumptions of Theorem 3.3, there exits a finite $t^0 \geq t_0$ such that, if $\sigma \in C([t_0,\infty) \to \mathbb{R}_+)$ satisfies (3.22) and* p *satisfies (3.19), then*

$$\mathrm{p}(t) \leq \|\mathrm{p}\|^{[t_0,t^0]} \exp\big(-\int_{t^0}^{t} \sigma(s)\mathrm{d}s\big) \text{ for } t \in [t^0, \infty). \tag{3.27}$$

Proof: By construction, $\mathrm{p}(t_0) \leq u(t_0)$. The proof of Theorem 3.4 is completed on showing that (3.23) and (3.26) imply that $\mathrm{p}(t) \leq u(t)$ for $t \geq t_0$. To that end, we invoke the function u_k above so that $\mathrm{p}(t) < u_k(t)$ for $t \in [t_0, t^0]$ and we establish that $p(t) \leq u_k(t)$ for $t \in [t^0, \infty)$.

Assume the contrary, so there is a value $t' \in (t^0, \infty)$ such that $\mathrm{p}(t') > u_k(t')$. Let $\mathcal{S}$ be the non-empty subset of $[t^0, \infty)$ such that $s \in \mathcal{S}$ implies $\mathrm{p}(s) \leq u_k(s)$. Let $t'' = \sup\{t \in S\}$ where $t'' < t'$ and $(t'', t'] \subset \mathcal{S}$; by continuity $\mathrm{p}(t'') = u_k(t'')$. Consequently, $\mathrm{D}^+\{p - u_k\}(t'') > 0$ while $p(t'') = u_k(t'')$. However, setting $t = t''$ in (3.26b) and (3.26d) and noting that $p(s) < u_k(s)$ for $s \in [t^0, t'']$ gives the result that $\mathrm{D}^+\mathrm{p}(t'') \geq u_k'(t'')$.

Thus our assumption is false (no such value t' exists) and hence $\mathrm{p}(t) < u_k(t)$ for every $t \in [t_0, \infty)$. Taking the limit as $k \to 1$ we deduce that $\mathrm{p}(t) \leq u(t)$ for every $t \in [t^0, \infty)$. ■

Theorem 3.4 throws the burden of finding the rate of decay of $\mathrm{p}(t)$ (asserted by (3.21b)) on the discovery of a suitable σ. A change of independent variable will in general yield a revised rate of decay.

Remark 3.1. *For $\tau(t) \in (0, \tau^+]$, $0 < \tau^+ < \infty$, a rate of exponential decay is provided in [19] from (3.27);* cf. *[31]. A concept of* generalized exponential stability *appears in the literature and one can exploit inequalities of the type (3.27). Theorem 3.7 can be applied to* $\mathrm{p}(t) = V(t, y(t))$ *(where V is a Lyapunov function – see Definition A.4 – and $y(t)$ is a solution of a DDE) to derive stability results. A definition of a Lyapunov* functional *arises if (*cf. *[12, 18]) $u \in \mathbb{R}^n$ is replaced by a function, and the arguments of μ,ν are related function norms (*e.g. $V(t, u) = \kappa_0\|u(t)\|^2 + \kappa_1 \int_{t-\tau(t)}^{t} \|u(s)\|^2 \mathrm{d}s$ *($\kappa_{0,1} \geq 0$, $\kappa_0 + \kappa_1 \neq 0$)).*

Suppose that the assumptions of Theorem 3.3 or 3.4 apply but consider in place of (3.19), the possibility of finding a *constant* $u(t) = \mathrm{p}_0 > 0$ satisfying $u'(t) \leq -\alpha(t)u(t)+\beta(t)\sup_{t-\tau(t)\leq s\leq t} u(s)+\varrho(t)$ $(t \geq t_0)$, with $\varrho \in BC[t_0, \infty)$ and $\varrho(t) \geq 0$, for $t \in [t_0, \infty)$. Taking $\mathrm{p}_0 = \sup_{t\geq t_0}\{\varrho(t)/(\alpha(t)-\beta(t))\}$ leads us to conclude the following corollary of Theorem 3.3 or 3.4.

Corollary 3.5. *Suppose the assumptions of Theorem 3.3, or 3.4, hold, and* p *satisfies (3.19) or (3.27), respectively. If $\varrho(t) \geq 0$ for $t \geq t_0$ and*

$$\mathrm{D}^+\mathrm{p}_\varrho(t)\leq-\alpha(t)\mathrm{p}_\varrho(t)+\beta(t)\sup_{t-\tau(t)\leq s\leq t}\mathrm{p}_\varrho(s)+\varrho(t) \quad (t \geq t_0) \tag{3.28}$$

then $\mathrm{p}_\varrho(t) \leq ||\mathrm{p}||^{[t_*,t_0]} + \sup_{t\geq t_0}\{\varrho(t)/(\alpha(t) - \beta(t))\}$.

Corollary 3.5, and its variants, play a rôle in investigations of stability w.r.t. a class of *persistent* perturbations (Definition A.2) where we consider

$$\widehat{y}'_+(t) = f(t, \widehat{y}(t), \widehat{y}(t - \tau(t)) + \delta f(t) \quad (t \in [t_0, \infty)), \tag{3.29a}$$

$$\widehat{y}(t) = \varphi(t) \quad (t \in [t_*, t_0]). \tag{3.29b}$$

3.6. A non-linear extension of Halanay's lemma, with unbounded lag

We rely on Assumption 1.1 and assume $w(t, 0, 0) \equiv 0$, and we suppose

$$\mathrm{D}^+\mathrm{p}(t) \leq w(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t)) \text{ for } t \geq t^0, \quad \overleftarrow{\mathrm{p}}(t^0) = \mathrm{p}^\natural \in [0, \infty). \tag{3.30}$$

Recall our notation and conditions: notably, $t - \tau(t) \to \infty$ as $t \to \infty$, $\overleftarrow{\mathrm{p}}(t) = \sup_{t-\tau(t)\leq s\leq t}\mathrm{p}(s)$. We give a result related to Proposition 1.2:

Theorem 3.6. *With our standard conditions, assume that $u_k \in C([t_*, \infty) \to \mathbb{R}_+)$, for $k \geq 1$, and*

$$u'_k(t) > w(t, u_k(t), \overleftarrow{u}_k(t)) \text{ for } t \geq t^0, \quad \overleftarrow{u}_k(t) = k \times \mathrm{p}^\natural \text{ for } t \in [t_*, t_0]. \tag{3.31}$$

Then $\mathrm{p}(t) < u_k(t)$ *for* $t \in [t_0, \infty)$ *if* $k > 1$, *while* $\mathrm{p}(t) \leq u_1(t)$ *for* $t \in [t_0, \infty)$.

Proof. Suppose the theorem is false. Let $t' = \inf_{s\geq t_0}\{\mathrm{p}(s) \geq u_k(s)\}$ so that $\mathrm{p}(t') = u_k(t')$. With $x(s) = \mathrm{p}(s) - u_k(s)$ there exists $t'' > t'$ such that $x[s_1, s_2] > 0$ for all $s_1 < s_2 \in [t', t'']$ and $\inf x[s_1, s_2] \geq 0$. However, $\mathrm{D}^+x(t) = \mathrm{D}^+\mathrm{p}(t) - u'_k(t)$ so $\mathrm{D}^+x(t') < 0$ and $\inf_{s\in[t',t'']} \mathrm{D}^+x(s) < 0$ and this is a contradiction. Thus $\mathrm{p}(t) < u_k(t)$ for $t \in [t_0, \infty)$ if $k > 1$ and on taking limits as $k \searrow 1$ the theorem follows. ■

By a modification of the above argument we can establish the following:

Theorem 3.7. *Suppose there exists $\eta^\natural$ such that, for any η with $0 < \eta \leq \eta^\natural$ and for all $t \geq t_0$. $w(t, \eta, \eta) \leq 0$. Then (3.30) implies that* $\mathrm{p}(t) \leq \mathrm{p}^\natural$ *(for $t \geq t_0$) when* $\mathrm{p}^\natural \leq \eta^\natural$.

Now consider the possibility of strengthening Theorem 3.7 to ensure decay of $\mathrm{p}(t)$. Baker & Tang discussed this issue under the condition $\mathrm{p}'(t) \leq w(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t))$ where $w(t, \eta, \eta) \leq -\varpi(\eta) < 0$. We prove a generalization. We suppose w (where $w(t, 0, 0) = 0$) satisfies Assumption 1.1 and the conditions of Theorem 3.7 hold with $\eta^{\natural}$ as there, but we strengthen our assumptions on w.

Lemma 3.8. *Assume that $w(t, u, v)$ is uniformly continuous for $u, v \in [0, \widehat{\eta^{\natural}}]$, $t \geq t_0$. Then, given $e > 0$, there exists $\delta > 0$ such that, if $u, v, \widehat{v} \in [0, \widehat{\eta^{\natural}}]$,*

$$w(t, u, v) - w(t, u, u) \leq e \text{ for all } t \geq t^0, \text{ and } u - v \leq \delta, \tag{3.32a}$$

$$|w(t, u, u) - w(t, \widehat{v}, \widehat{v})| \leq e \text{ for all } t \geq t^0 \text{ and } u - \widehat{v} \leq \delta. \tag{3.32b}$$

Theorem 3.9. *With the assumptions of Theorem 3.7, suppose*

$$w(t, \eta, \eta) \leq -\varpi(\eta) < 0 \text{ for any } \eta \text{ with } 0 < \eta \leq \widehat{\eta^{\natural}} \text{ and for all } t \geq t_0. \tag{3.33}$$

with $\widehat{\eta^{\natural}} = \kappa\eta^{\natural}$ where $\kappa > 1$. Also, suppose that, (3.32) holds. Then, if $\mathrm{p} \in C([t_0, \infty) \to \mathbb{R}_+)$ and $\mathrm{D}^+\mathrm{p}(t) \leq w(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t))$ for $t \in [t^0, \infty)$ it follows that $\mathrm{p}(t) \to 0$ as $t \to \infty$.

Proof. Suppose that $\mathrm{p}^{\natural} \leq \eta^{\natural}$ such that Theorem 3.7 holds and we have $\eta^{\natural} \geq \mathrm{p}^{\natural} \geq \widehat{\mathrm{p}}$ where $\widehat{\mathrm{p}} := \limsup_{t\to\infty} \mathrm{p}(t) = \limsup_{t\to\infty} \overleftarrow{\mathrm{p}}(t) \geq 0$ and the condition $\mathrm{p}(t) \to 0$ is equivalent to the condition $\widehat{\mathrm{p}} = 0$. We assume instead that $\widehat{\mathrm{p}} > 0$ and we obtain a contradiction.

Given $\delta \in (0, (\kappa - 1)\eta^{\natural}]$, there exists $t^{\delta} \in [t^0, \infty)$ such that $\overleftarrow{\mathrm{p}}(s) \leq \widehat{\mathrm{p}} + \delta$ for $s \in [t^{\delta}, \infty)$. With the same $\delta > 0$ there exists an unbounded set $\mathcal{S}(\delta) \subseteq [t^{\delta}, \infty)$ such that $t \in \mathcal{S}(\delta)$ implies $\mathrm{p}(t) > \widehat{\mathrm{p}} - \delta$. Thus, for $t \in \mathcal{S}(\delta)$, $\overleftarrow{\mathrm{p}}(t) - \widehat{\mathrm{p}} \leq \delta$, $\widehat{\mathrm{p}} - \mathrm{p}(t) \leq \delta$. By (3.32), if we pick $e = \frac{1}{4}\varpi(\widehat{\mathrm{p}})$, $\delta = \delta(e)$ can be chosen so that $|w\big(t, \mathrm{p}(t), \mathrm{p}(t)\big) - w\big(t, \widehat{\mathrm{p}}, \widehat{\mathrm{p}}\big)| < \frac{1}{4}\varpi(\widehat{\mathrm{p}})$, $w\big(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t)\big) - w\big(t, \mathrm{p}(t), \mathrm{p}(t)\big) < \frac{1}{4}\varpi(\widehat{\mathrm{p}})$. Thus, (3.30) implies that (for $t \in \mathcal{S}(\delta)$)

$$\mathrm{D}^+\mathrm{p}(t) \leq w(t, \mathrm{p}(t), \mathrm{p}(t)) + w(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t)) - w(t, \mathrm{p}(t), \mathrm{p}(t)) < -\frac{1}{2}\varpi(\widehat{\mathrm{p}}). \tag{3.34}$$

Now suppose that $t^{[1]} \in \mathcal{S}(\frac{1}{2}\delta)$. Then (as p is continuous) there exists $t^{[2]} > t^{[1]}$ with $[t^{[1]}, t^{[2]}] \subseteq \mathcal{S}(\delta)$, and (by (3.34))

$$\mathrm{p}[t^{[1]}, t^{[2]}] \leq \sup\nolimits_{t^{[1]} \leq s \leq t^{[2]}} \mathrm{D}^+\mathrm{p}(s) < -\tfrac{1}{2}\varpi(\widehat{p}).$$

Thus $\mathrm{p}(t^{[2]}) < \mathrm{p}(t^{[1]})$ and with an appropriate choice of $t^{[2]}$, $\mathrm{p}(t^{[2]}) = \widehat{p} - \delta$. Now, by the definition of $\widehat{\mathrm{p}}$ and the continuity of p there exist $t^{[3]} < t^{[4]}$ with $t^{[2]} \leq t^{[3]}$ such that $\widehat{p} - \delta < \mathrm{p}(s) < \widehat{p} - \frac{1}{2}\delta$ for $s \in (t^{[3]}, t^{[4]})$, and $\mathrm{p}(t^{[3]}) < \mathrm{p}(t^{[4]}) = \widehat{\mathrm{p}} - \frac{1}{2}\delta$. It follows, since $[t^{[3]}, t^{[4]}] \subseteq S(\delta)$, that

$$\mathrm{p}[t^{[3]}, t^{[4]}] \leq \sup\nolimits_{t^{[3]} \leq s \leq t^{[4]}} \mathrm{D}^+\mathrm{p}(s) < -\tfrac{1}{2}\varpi(\widehat{p}).$$

which contradicts the property $\mathrm{p}(t^{[4]}) > \mathrm{p}(t^{[3]})$. Our assumption that $\widehat{\mathrm{p}} > 0$ is therefore false; $\widehat{\mathrm{p}} = 0$ and the theorem follows. ■

An alternative approach is to decompose $w(t, u, v)$ into the form $A(t)u + B(t)v + \delta w(t, u, v)$ where $\delta w(t, u, v) = \mathcal{O}(u^2 + uv + v^2)$ as $u, v \to 0$, in parallel with discussions of stability in the first approximation. We shall not pursue this. Yet a further approach is to seek a positive monotone increasing and differentiable function $\mathrm{r}(t)$ for which $\mathrm{p_r}(t) := \mathrm{r}(t)\mathrm{p}(t)$ can be shown to be bounded, using Theorem 3.7. As $\mathrm{p}(t) = \mathrm{p_r}(t)/\mathrm{r}(t)$, $\overleftarrow{\mathrm{p}(t)} \leq \overleftarrow{\mathrm{p_r}}(t) \sup_{s\in(t-\tau(t),t)}[1/\mathrm{r}(t)] = \overleftarrow{\mathrm{p_r}}(t)[1/\mathrm{r}(t - \tau(t))]$ and, by assumption, $\mathrm{D}^+\{\mathrm{r}(t)\mathrm{p}(t)\} = \mathrm{r}(t)\mathrm{D}^+\mathrm{p}(t) + \mathrm{r}'(t)\mathrm{p}(t)$.

Choosing $\mathrm{r}(t) = \exp(\gamma_+ t)$ gives $\mathrm{D}^+\mathrm{p}_\mathrm{r}(t) = \{\mathrm{D}^+\mathrm{p}(t) + \gamma_+ \mathrm{p}(t)\} \exp(\gamma_+ t)$ and

$$\mathrm{D}^+\mathrm{p}_\mathrm{r}(t) \leq \exp\{\gamma_+ t\} w\big(t, \mathrm{p}(t), \overleftarrow{\mathrm{p}}(t)\big) + \gamma_+ \exp\{\gamma_+ t\}\mathrm{p}(t) \tag{3.35}$$

(invoking (3.30)). If we define

$$w_t(t, u, v) = \exp\{\gamma_+ t\} w\big(t,\, u\exp\{\gamma_+ t\},\, v\exp\{\gamma_+(t-\tau(t))\}\big) \tag{3.36}$$

then $\mathrm{D}^+\mathrm{p}_\mathrm{r}(t) \leq w_\mathrm{r}(t,\, \mathrm{p}_\mathrm{r}(t),\, \overleftarrow{\mathrm{p}_\mathrm{r}}(t))$ and we require $w_\mathrm{r}(t, u, u) \leq 0$ for sufficiently small u to appeal to Theorem 3.7, with p_r in place of p, and establish:

Corollary 3.10. *Suppose that (3.30) is satisfied and, for some $\gamma_+ > 0$, there exists $\eta^\natural$ such that, for any η with $0 < \eta \leq \eta^\natural$*

$$w\big(t,\, \eta\exp\{-\gamma_+ t\},\, \eta\exp\{-\gamma_+(t-\tau(t))\}\big) \leq -\gamma_+\eta < 0 \text{ for } t \geq t_0\ . \tag{3.37}$$

Then $\mathrm{p}(t)\exp\{\gamma_+ t\}$ is bounded for $t \in [t_0, \infty)$, provided $\mathrm{p}^\natural$ is sufficiently small.

We introduced t^0 with (here) arbitrary $t^0 \geq t_0$ as conditions may be satisfied on $[t^0, \infty)$ but not on $[t_0, \infty)$. The results show that sufficiently small initial conditions on p imply boundedness (or decay) of p under appropriate conditions. Compare Theorem 3.7 with [18, Proposition 2]. A typical form for $w(t, x, y)$ is $w_1(t, x) + w_2(t, y)$. Halanay's original lemma corresponds to $t^0 = t_0$ and the case $\tau(t) = \tau^+ > 0$ (and in Theorem 3.7 the choice $w(t, x, y) = -ax + by$ where $a \geq b > 0$ or $a > b > 0$, and arbitrary $\eta^\natural > 0$).

Ex.3.4. *Consider the problem*

$$y'(t) = \lambda(t)y(t)\{1 - y(t-\tau(t))\}, \quad (t \geq t_0), \quad y(t) = \varphi(t) \quad (t \leq t_0), \tag{3.38}$$

where $\inf\{\tau(t)\} > 0$, $\lim_{t\to\infty} t - \tau(t) = \infty$, $\inf\{\lambda(t)\} > 0$, $\sup\{\lambda(t)\} < \infty$ *and* $\sup\{\lambda(t)\}^2\tau(t) < \infty$. *If $\varphi_0(t) = 0$ for $t \leq t_0$, $\varphi_1(t) = 1$ for $t \leq t_0$, then $y(\varphi_0; t) = 0$ and $y(\varphi_1; t) = 1$ for $t \geq t_0$ We find a condition on $\lambda(t)$, $\tau(t)$ that ensures stability of the positive steady state solution $y(t) \equiv y(\varphi_1; t) = 1$. Our standard results apply to the zero solution of a DDE, so we write $y(\varphi_1; t) = 1 + u(t)$ and, equivalently, consider stability of the solution $u(\varphi_0; t) \equiv 0$ of $u'(t) = -\lambda(t)\{1 + u(t)\}u(t - \tau(t))$. Rearranging,*

$$u'(t) = -\lambda(t)u(t-\tau(t)) - \lambda(t)u(t)u(t-\tau(t)) \tag{3.39}$$

Routinely, $u(t-\tau(t)) = u(t) - \tau(t)u'(t - \widehat{\tau}(t))$ where $\widehat{\tau}(t) \in (0, \tau(t))$. We set $\sigma(t) = t - \widehat{\tau}(t)$; (3.39) yields $u'(\sigma(t))$ on replacing t by $\sigma(t)$. Thus,

$$\begin{aligned} u'(t) \quad &= \quad -\lambda(t)u(t) + \lambda(t)^2\tau(t)u(\sigma(t)) \\ &+ \quad \lambda(t)^2\tau(t)u(\sigma(t))u(\sigma(t) - \tau(\sigma(t))) - \big\{\lambda(t)u(t)u(t-\tau(t))\big\}, \end{aligned} \tag{3.40}$$

where $t - \tau(t) \leq \sigma(t) < t$ and $t - 2\tau(t) \leq \sigma(t) - \tau(\sigma(t)) < t$. Choose the Lyapunov function as $V(u) = \frac{1}{2}u^2$ (with $\mu(s) = \nu(s) = \frac{1}{2}s^2$). For the solution of (3.39), we have $V'(u(t)) = u(t)u'(t)$ so

$$\begin{aligned} V'(u(t)) \quad &= \quad -\lambda(t)u^2(t) + \lambda(t)^2\tau(t)u(t)u(t-\tau(t)) \\ &+ \quad \lambda(t)^2\tau(t)u(t)u(\sigma(t))u(\sigma(t) - \tau(\sigma(t))) - \big\{\lambda(t)u^2(t)u(t-\tau(t))\big\} \end{aligned} \tag{3.41}$$

for all $t \geq t_0$. As $\lambda(t) > 0$, (3.41) gives

$$V'(u(t)) \quad \leq \quad -\lambda(t)V(t) + \lambda(t)^2\tau(t)\overleftarrow{V}(t) + (\lambda(t) + \lambda(t)^2\tau(t))\overleftarrow{V}^{3/2}(t)$$

for $t \geq t_0$, where $\overleftarrow{V}(t) = \max_{\sigma(t)-\tau(\sigma(t))\leq s\leq t} V(s)$. Define

$$w(t, v_1, v_2) = -\lambda(t)v_1 + \lambda(t)^2\tau(t)v_2 + (\lambda(t) + \lambda(t)^2\tau(t))v_2^{3/2}. \tag{3.42}$$

If $\lambda(t)\tau(t) < 1$, and $0 < \eta < \{(1 - \lambda(t)\tau(t))/2(1 + \lambda(t)\tau(t))\}^2$, we have $w(t, \eta, \eta) = \{(-\lambda(t) + \lambda(t)^2\tau(t)) + (\lambda(t) + \lambda(t)^2\tau(t))\eta^{1/2}\}\eta$ so $w(t, \eta, \eta) < \lambda(t)(-1 + \lambda(t)\tau(t))\eta/2 < 0$ for all $t \geq t_0$. By Theorem 3.9, the solution $u(t) \equiv 0$ of (3.39) is asymptotically stable. Compare the exposition in [15, pp.55 et seq.]. Note how we overcame the absence of a linear term in $u(t)$ in (3.39).

3.7. **Finite-difference and discrete-continuous analogues**

We have encountered linear and nonlinear relationships between (say) $\widetilde{y}_{n+1}$ and $\{\widetilde{y}_\ell\}_{n-N(n)}^n$, or between $\widetilde{y}(t_{n+1})$, $\widetilde{y}(t_n)$ and $\{\widetilde{y}_{\ell+s_\ell}\}_{n-N(n)}^n$ (or possibly between $\widetilde{y}(t_{n+s})$, $\widetilde{y}(t_n)$ and $\{\widetilde{y}_{\ell+s_\ell}\}_{n-N(n)}^n$ where $s, s_\ell \in [0, 1)$). We generally have a recurrence that can be expressed using an *increment function* defining $\widetilde{y}(t_{n+1}) - \widetilde{y}(t_n)$ or $\widetilde{y}_{n+1} - \widetilde{y}_n$. If we wish to analyze stability (definitions of which are obvious analogues of those in Definitions A.1, A.3), we can do so via a discretely-defined or classical Lyapunov function(al), that defines a sequence $\{\mathrm{p}_n\}$ or a densely-defined function $\mathrm{p}(t)$. Discrete values $\widetilde{y}(t_\ell)$ generate positive sequences $\{\mathrm{p}_n\}$ – *e.g.* $\mathrm{p}_n = \|\widetilde{y}(t_\ell)\|$; more generally, using a Lyapunov function, $\mathrm{p}_n = V(t_n, \widetilde{y}(t_n))$ – to discuss qualitative behaviour. For a densely-defined function p, we can sometimes generate an inequality for $(\frac{\mathrm{d}}{\mathrm{d}t})_+\mathrm{p}(t_{n+s})$.

Assumption 3.1. *(a) With $\mathcal{T} := \{t_0, t_1, t_2, \cdots\}$, $h_n := t_{n+1} - t_n$ ($n \in \mathbb{Z}_+$), we assume $h_n \in [h_\star, h^\star] \subset (0, \infty)$. (b) We suppose the positive sequence $\{\mathrm{p}_n\}$ is defined for $n \geq n_* \in \mathbb{Z}$ and $n_0 > n_*$ is such that for $n \geq n_0$ there exists $N(n) \in \mathbb{Z}$ with $n_* = \min_{n\geq n_0}\{n - N(n)\}$. (c) We suppose $n - N(n) \to \infty$ as $n \to \infty$.*

As examples of discrete inequalities, we might consider

$$\mathrm{p}_{n+1} - \mathrm{p}_n \leq W_{\mathcal{T}}\big(n;\ \mathrm{p}_n, \overleftarrow{\mathrm{p}_{n-1}}\big),\ \mathrm{p}_{n+1} - \mathrm{p}_n \leq W_{\mathcal{T}}\big(n;\ \mathrm{p}_{n+1}, \overleftarrow{\mathrm{p}_n}\big), \tag{3.43a}$$

or

$$\mathrm{p}_{n+1} - \mathrm{p}_n \leq W_{\mathcal{T}}^{\natural}\big(n;\ \mathrm{p}_n\ ,\ \overleftarrow{\mathrm{p}_n}\big) \quad (n \in \mathbb{Z}_+). \tag{3.43b}$$

We can term the right-hand sides in (3.43) *increment bounds*. The natural analogue with the continuous case is preserved if, *e.g.*, we replace (3.43b) by

$$\mathrm{p}[\,t_n, \mathrm{p}(t_{n+1})] = w_{\mathcal{T}}^{\natural}\big(t_n;\ \mathrm{p}(t_n), \overleftarrow{\mathrm{p}}(t_n)\big) \quad (n \in \mathbb{Z}_+). \tag{3.43c}$$

We can also consider inequalities similar to (3.43) that bound $\mathrm{p}[\,t_n, t_{n+s}]$ or $(\frac{\mathrm{d}}{\mathrm{d}t})_+\mathrm{p}(t_{n+s})$ ($s \in [0, 1]$). For each of these and related variants one may seek discrete or discrete-continuous variants and extensions of Halanay's Lemma, but space limitations curtail the extent of our presentation here.

We start with a refinement of the result in Example 1.3. Having previously defined $\overleftarrow{\mathrm{p}}(t) = \sup_{t-\tau(t)\leq s\leq t} \mathrm{p}(s)$ we now assume $N(n) \in \mathbb{Z}_+$ for $n \in \mathbb{Z}_+$, $n - N(n) \to \infty$ as $n \to \infty$ and define

$$\overleftarrow{\mathrm{p}_n} := \max_{n-N(n)\leq\ell\leq n}\{\mathrm{p}_\ell\}. \tag{3.44}$$

The reader should distinguish between $\overleftarrow{\mathrm{p}}(t_n)$ and $\overleftarrow{\mathrm{p}_n}$.

Theorem 3.11. *Let* $\{\mathrm{p}_\ell\}$ *be a sequence of positive values that satisfy*

$$\{\mathrm{p}_{n+1} - \mathrm{p}_n\} \le -h_n A_n \mathrm{p}_n + h_n B_n \overleftarrow{\mathrm{p}_n} \ \textit{for}\ n \in \{0, 1, 2, \cdots\}, \tag{3.45}$$

where $h_n \in [h_*, h^*] \subset (0, \infty)$*, and* $0 < h_n A_n < 1$*,* $0 \le B_n \le A_n$ *then* $\mathrm{p}_n \le \overleftarrow{\mathrm{p}_0}$*.*

If $\{\mathrm{r}_\ell\}$ is positive and increasing (*e.g.*, $\mathrm{r}_n = \nu^n$ with $\nu > 1$) and the above result can be applied to $\{\mathrm{p}_n^{[\mathrm{r}]}\}$, with $\mathrm{p}_\ell^{[\mathrm{r}]} := \mathrm{r}_\ell \mathrm{p}_\ell$ (instead of to $\{\mathrm{p}_n\}$) a rate of decay for $\{\mathrm{p}_n\}$ follows.

Assumption 3.2. $t_{\ell_s(n)} \in \{t_0, t_1, ..., t_n\}$ *for* $s \in \mathbb{N}$*,* $t_{\ell_s(n)} \to \infty$ *as* $n \to \infty$ *The functions* $a(t) = a(\mathcal{T}; t)$*,* $b(t) = b(\mathcal{T}; t)$ *are assumed to be continuous in* t *and bounded for* $t \in \mathbb{R}_+$*.*

Theorem 3.12. *Suppose the positive continuous function* $\mathrm{p}(t)$ *satisfies, with Assumption 3.2,*

$$\mathrm{p}(t_{n+1}) - \mathrm{p}(t_n) \le -a(t_n)\mathrm{p}(t_n) + \sum_{s=1}^{n} b(t_s)\mathrm{p}(t_{\ell_s(n)}), \ \textit{for all}\ t_n \in \mathcal{T}, \tag{3.46a}$$

$$\mathrm{p}(t) \le \overleftarrow{\mathrm{p}}(t_0), \qquad (\textit{for all}\ t \le t_0) \tag{3.46b}$$

and $\beta(t_n) = \sum_{s=n-N(n)}^{n} b(t_s)$*. If there exist constants* $\check{a}$ *and* $c \in (0, 1)$ *such that*

$$0 < \check{a} \le a(t_n) \le \hat{a} < 1, \ \textit{and}\ -a(t_n) + \beta(t_n) \le -c < 0 \tag{3.47}$$

uniformly for all $n \in \mathbb{Z}_+$*, then*

$$\mathrm{p}(t_n) \le M||\mathrm{p}||^{(-\infty, t_0]} \quad \textit{and} \quad \lim_{n\to\infty} \mathrm{p}(t_n) = 0. \tag{3.48}$$

Proof. For any $k > 1$, $\mathrm{p}(t) < k\overleftarrow{\mathrm{p}}(t_0) = k\|\mathrm{p}\|^{(-\infty, t_0]}$ and we shall deduce that

$$\mathrm{p}(t_n) < k\overleftarrow{\mathrm{p}}(t_0) \text{ for } n \ge 0. \tag{3.49}$$

Clearly, $\mathrm{p}(t_0) < k\overleftarrow{\mathrm{p}}(t_0)$. If (3.49) is not true, there exists $n_0 > 0$ such that

$$\mathrm{p}(t_n) < k\overleftarrow{\mathrm{p}}(t_0) \text{ for } 0 < n < n_0 \ \text{ and } \ \mathrm{p}(t_{n_0}) \ge k\overleftarrow{\mathrm{p}}(t_0). \tag{3.50}$$

Now either (a) $\mathrm{p}(t_{n_0+1}) > \mathrm{p}(t_{n_0})$ or (b) $\mathrm{p}(t_{n_0+1}) \le \mathrm{p}(t_{n_0})$. In the first case, (a), $\mathrm{p}(t_{n_0+1}) - \mathrm{p}(t_{n_0}) > 0$. However, from (3.47),

$$\begin{aligned}\mathrm{p}(t_{n_0+1}) - \mathrm{p}(t_{n_0}) &\le -a(t_{n_0})\mathrm{p}(t_{n_0}) + \sum_{s=n_0-N(n_0)}^{n_0} b(t_s)\mathrm{p}(t_{\ell_s(n)}) \\ \le \{-a(t_{n_0}) + \sum_{s=n_0-N(n_0)}^{n_0} b(t_s)\mathrm{p}(t_{\ell_s(n)})\} &\times k\overleftarrow{\mathrm{p}}(t_0) \le -ck\overleftarrow{\mathrm{p}}(t_0) < 0,\end{aligned}$$

which is a contradiction. In the second case, (b), $\mathrm{p}(t_{n_0+1}) \le \mathrm{p}(t_{n_0})$. Using the assumption, $\beta(t_n) < a(t_n) \le 1$, $\mathrm{p}(t_{n_0+1}) - \mathrm{p}(t_{n_0-1}) = \mathrm{p}(t_{n_0+1}) + \{\mathrm{p}(t_{n_0}) - \mathrm{p}(t_{n_0-1})\} - \mathrm{p}(t_{n_0})$ gives

$$\mathrm{p}(t_{n_0+1}) - \mathrm{p}(t_{n_0-1}) \le \mathrm{p}(t_{n_0+1}) - a(t_{n_0-1})\mathrm{p}(t_{n_0-1}) - (1 - \beta(t_{n_0-1}))\mathrm{p}(t_{n_0}),$$

so $-(1 - a(t_{n_0-1}))\mathrm{p}(t_{n_0-1}) \le -(1 - \beta(t_{n_0-1}))\mathrm{p}(t_{n_0})$. It follows that $(1 - \beta(t_{n_0-1}))\mathrm{p}(t_{n_0}) \le (1 - a(t_{n_0-1}))\mathrm{p}(t_{n_0-1}) < (1 - \beta(t_{n_0-1}))\mathrm{p}(t_{n_0-1})$, which yields $\mathrm{p}(t_{n_0}) < \mathrm{p}(t_{n_0-1})$, which contradicts our assumption on n_0. Consequently, the inequality (3.49) must hold. Letting $k \searrow 1$, we conclude that the first part of (3.48) holds for all $t_n \in \mathcal{T}$.

We now prove the second part of (3.48). Let lim $\sup_{n\to\infty} \mathrm{p}(t_n) = l$, then $0 \le l \le \overleftarrow{\mathrm{p}}(t_0)$. We suppose $l > 0$ and obtain a contradiction. Introduce $A = \limsup_{n\to\infty}\{1 + a(t_n)\}$, $B = \limsup_{n\to\infty} 1+\sum_{s=n-N(n)}^{n} b(t_s)$; clearly $\eta^\natural := \{A-B\}/\{A+B\} \in (0, 1)$. For arbitrary $\eta \in (0, \eta^\natural)$, by the definition of l and the properties of superior limits, there exists some unbounded subsequence $\{t_{n_k}\}$ of $\mathcal{T}$ and a sufficiently large integer $L > 0$ such that

$$l - \eta < \mathrm{p}(t_{n_k}) < l + \eta,\ \mathrm{p}(t_{\ell_s(n)}) < l + \eta \text{ for all } t_{n_k} \ge t_L \text{ and } t_{\ell_s(n)} \ge t_L.$$

Rearranging the inequality in (3.46), we deduce

$$\begin{aligned} \{1 + a(t_{n_k})\}(l - \eta) &< \mathrm{p}(t_{n_k+1}) + \{1 + a(t_{n_k})\}\mathrm{p}(t_{n_k}) \\ \mathrm{p}(t_{n_k}) + \sum_{s=n-N(n_k)}^{n_k} b(t_{n_s})\mathrm{p}(t_{\ell(s)}) &< \{1 + \sum_{s=n-N(n)}^{n} b(t_s)\}(l + \eta) \end{aligned} \tag{3.51}$$

for all $t_{n_k} \ge t_N$, $(t_{\ell_s} \ge t_N)$. It follows on taking limits that $\eta > (A-B)l/(A+B)$. This contradicts the choice of η. Hence $l = 0$, and the theorem follows. ■

Theorem 3.12 is related to similar results for inequalities $\mathrm{p}(t_{n+1}) - \mathrm{p}(t_n) \le -a(t_n)\mathrm{p}(t_n) + \beta(t_n)\overleftarrow{\mathrm{p}}(t_n)$, and (*cf.* Theorem 3.11)

$$\mathrm{p}_{n+1} - \mathrm{p}_n \le -a_n\mathrm{p}_n + \beta_n\overleftarrow{\mathrm{p}_n}. \tag{3.52}$$

Our next result, whose simplicity is worthy of note, relates to a nonlinear inequality.

Theorem 3.13. *Suppose $\omega(n, u, v)$ is monotonic decreasing in u, and monotonic non-decreasing in v, with $\omega(n, 0, 0) = 0$ (for every integer $n \ge n_0$) and there exists $\eta^\natural$ such that $\omega(n, u, u) \le 0$ for all $n \ge n_0$, if $0 < u \le \eta^\natural$. Suppose the positive sequence $\{\mathrm{p}_n\}$ satisfies $\mathrm{p}_{n+1} \le \mathrm{p}_n + \omega\big(n, \mathrm{p}_n, \overleftarrow{\mathrm{p}_n}\big)$ for $n \ge n_0$. Then, provided $\overleftarrow{\mathrm{p}_{n_0}} \le \eta^\natural$, it follows that $\mathrm{p}_n \le \overleftarrow{\mathrm{p}_{n_0}}$ for all $n \ge n_0$.*

Proof. Suppose $\overleftarrow{\mathrm{p}_{n_0}} \le \eta^\natural$; clearly $\mathrm{p}_{n_0} \le \overleftarrow{\mathrm{p}_{n_0}}$, $\omega(n_0, \overleftarrow{\mathrm{p}_{n_0}}, \overleftarrow{\mathrm{p}_{n_0}}) \le 0$ by assumption, and
$$\mathrm{p}_{n_0+1} \le \mathrm{p}_{n_0} + \omega(n_0, \mathrm{p}_{n_0}, \overleftarrow{\mathrm{p}_{n_0}}) \le \mathrm{p}_{n_0} + \omega(n_0, \overleftarrow{\mathrm{p}_{n_0}}, \overleftarrow{\mathrm{p}_{n_0}}) + \{\omega(n_0, \mathrm{p}_{n_0}, \overleftarrow{\mathrm{p}_{n_0}}) - \omega(n_0, \overleftarrow{\mathrm{p}_{n_0}}, \overleftarrow{\mathrm{p}_{n_0}})\},$$

the term in $\{\ \}$ is non-positive, and $\mathrm{p}_{n_0+1} \le \mathrm{p}_{n_0} \le \eta^\natural$. Now, $\mathrm{p}_{n_0+1} \le \overleftarrow{\mathrm{p}_{n_0+1}} \le \eta^\natural$; the theorem follows by induction. ■

A natural generalization of (3.52) arises on considering $\mathrm{p}_{n+1} - \mathrm{p}_n \le -a_n\omega_1(\mathrm{p}_n) + b_n\omega_2(\overleftarrow{\mathrm{p}_n})$, where $\omega_{1,2}$ are wedge functions. Baker & Tang have discussed such results.

4. EPILOGUE

This paper has been reduced to two-thirds the length of an earlier draft and, with so much material omitted, we shall return to our subject elsewhere [3]. In particular we shall expand on discrete analogues of §§3.2, 3.3 and 3.7. Some of our present results derive from earlier work performed by the author with Arsalang Tang. The literature contains a number of results – sometimes restatements, or re-discoveries – which in some sense ensue (are derived, or adapted) from Halanay's theory or from various techniques that establish this theory. Tang's thesis [30], associated Technical Reports of the (now defunct) Manchester Centre for Computational Mathematics, on-line working papers (*cf.* CiteSeer) and subsequent papers, merit further consultation by those interested.

The author thought to include an extensive bibliography but, space being restricted, concluded that a limited set of citations together with reference to MathSciNet, Zentralblatt MATH, and Google Scholar, should suffice. The IEEE journals (e.g., IEEE Transactions on Automatic Control) prove a rich source that should not be overlooked.

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Appendix

A. Standard definitions

In this section we collect some standard definitions employed in our text.

Definition A.1. *(*Stability w.r.t. perturbed initial conditions*) Suppose $f(t,0,0)\equiv 0$; then the zero solution of (2.1) is* (i) stable *(w.r.t. perturbed initial conditions) if for each $\epsilon > 0$ there exists a corresponding $\delta_0 = \delta_0(\epsilon, t_0)$ such that, for any initial function $\varphi \in BC[t_*, t_0]$ with $\|\varphi\| < \delta_0$, $\|y(t)\| < \epsilon$ $(t \geq t_0)$;* (ii) asymptotically stable *if, in addition, there exists $\delta_1 \leq \delta_0$ such that $y(t) \to 0$ as $t \to \infty$ for any initial function $\varphi \in BC[t_*, t_0]$ with $\|\varphi\| < \delta_1$;* (iii) exponentially stable *if , in addition, there exists $\delta_2 \leq \delta_0$ such that $y(t) \to 0$ exponentially as $t \to \infty$ for any initial function $\varphi \in BC[t_*, t_0]$ with $\|\varphi\| < \delta_2$. (iv) If δ_ℓ is independent of t_0 the stability is called* uniform. *(v) A solution that is not stable is* unstable.

Definition A.2. *(*Stability w.r.t. bounded persistent perturbations*) A solution y of the equations (2.1) is* stable *(w.r.t. bounded persistent perturbations) if for each $\epsilon > 0$ there exists a corresponding $\Delta = \Delta(\epsilon, t_0)$) such that, for any function $\delta f \in BC[t_0, \infty)$ with $\|\delta f\|^{(t_0,\infty)} < \Delta$, $\|\widehat{y}(t) - y(t)\| < \epsilon$ for all $t \geq t_0$.*

Definition A.3. *The zero solution of the equations (2.15) or (2.16) is* (i) stable *if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for any initial function $\phi \in BC^1[t_*, t_0]$, with $\|\phi\| < \delta$, $|\widetilde{y}_\Theta(\phi, \mathcal{T}; t)| < \epsilon$ $(t \geq t_0)$;* (ii) uniformly stable *if the number δ in definition (i) is independent of t_0;* (iii) uniformly asymptotically stable *if it is uniformly stable and there exists a constant δ_0 such that $\widetilde{y}(t) \equiv \widetilde{y}_\Theta(\phi, \mathcal{T}; t) \to 0$ as $t \to \infty$ for any initial function $\phi \in BC^1[t_*, t_0]$ with $\|\phi\| < \delta_0$.*

Definition A.4. *(*Lyapunov functions*) Assume that the functions $\nu(s)$ and $\mu(s)$ are* wedge functions *(continuous, positive and increasing for $s > 0$, with $\nu(0) = \mu(0) = 0$ and $\mu(s), \nu(s) \to \infty$ as $s \to \infty$). By assumption, μ and ν have inverse functions μ^{-1}, ν^{-1} on $[0,\infty)$. $V : [t^0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ is called a* Lyapunov function *if it is a positive definitive continuous function satisfying $\mu(\|u\|) \leq V(t,u) \leq \nu(\|u\|)$.*

Definition A.5. *(see [29]) Suppose that $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$, and also the induced subordinate norm on matrices in $\mathbb{R}^{n\times n}$. For $A \in \mathbb{R}^{n\times n}$, we denote by $\mu_{\|\cdot\|}(A)$ the* (Lozinskii) measure *or* logarithmic norm *of A w.r.t. $\|\cdot\|$ where*

$$\mu_{\|\cdot\|}(A) := \mathrm{D}^+\|I + \xi A\|\Big|_{\xi=0} = \lim_{\delta\xi\to 0} \frac{\|I + |\delta\xi| \times A\| - 1}{|\delta\xi|} \in (-\infty, \infty). \tag{A.1}$$